Relative Error of the Generalized Pareto Approximation to Value-at-Risk

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Abstract

The relative error of the approximation to Value-at-Risk is evaluated when the conditional tail distribution is approximated by a generalized Pareto distribution for a fixed threshold. The main concern is the uniformity of the convergence to zero as the threshold tends to infinity. It is proved that the convergence is uniform if the underlying distribution is in the Fréchet class or Weibull-type with index 1 in the Gumbel class where the slowly varying factor has a positive constant as a limit. Otherwise it is not uniform without limiting the range of values into an interval depending on the threshold. A good example is the case of the t distribution. The relative approximation error uniformly converges to zero since it belongs to the Fréchet class and the slowly varying factor has a positive limiting constant. But it does not hold true in the case of the normal distribution, although it is the limit of the t distribution when the degree of freedom tend to infinity. This example indicates that the approximation error is fragile and caution is necessary when the Value-at-Risk is approximated by a generalized Pareto distribution as is in the peaks-over-threshold (POT) method.

1 Introduction

One of the main objectives of financial risk management is to estimate a Value-at-Risk (VaR); a high quantile $q_\alpha := F^{-1}(\alpha)$ of a loss distribution F. In recent years, a method which applies extreme value theory, especially the so-called peaks-over-threshold (POT) method, has attracted much attention in the VaR estimation literature. This method is based on theorems derived in Pickands (1975) and Balkema and de Haan (1974), stating that the tails of almost all common distributions over some high threshold u can be approximated by the generalized Pareto distribution. A weakness of their approach is that although the approximation of the distribution function is guaranteed, that of the quantile or VaR, is not guaranteed.

In general, we can write

$$F(x) = F(u) + F_u(x-u)(1-F(u)),$$

for $x > u$, where $F_u(x) = (F(x+u)-F(u))/(1-F(u))$. The generalized Pareto approximation F_G of F for a given threshold $u > 0$ and $x > u$ is given by

$$F_G(x) = F(u) + G_{\xi,\sigma(u)}(x-u)(1-F(u)),$$

where $G_{\xi,\sigma}(x)$ is the generalized Pareto distribution function defined as

$$G_{\xi,\sigma}(x) = \begin{cases} 1 - (1 + \xi x/\sigma)^{-1/\xi}, & \xi > 0, \\ 1 - \exp(-x/\sigma), & \xi = 0. \end{cases}$$

Such an approximation is justified as far as F belongs to the maximum domain of attraction (of the generalized extreme value distribution) with tail index ξ (see Pickands (1975) or Balkema and de Haan (1974)). If a distribution function F is twice differentiable, $f(x) = F'(x) > 0$ and

$$\lim_{x\to\infty} \frac{d\phi(x)}{dx} = \xi,$$

where $\phi(x) = (1 - F(x))/f(x)$, then F belongs to the maximum domain of attraction with tail index ξ. This sufficient condition is the so-called von Mises' condition (see, de Haan and Ferreira (2006)). We assume the condition through this paper. We call the distributions which can be approximated by $G_{\xi,\sigma}$ with $\xi > 0$ as the Fréchet class and that by $G_{\xi,\sigma}$ with $\xi = 0$ as the Gumbel class.

An approximated quantile $\widetilde{q}_\alpha = F_G^{-1}(\alpha)$ is given by

$$\widetilde{q}_\alpha = \begin{cases} u + \dfrac{\sigma(u)}{\xi}\left\{\left(\dfrac{1-\alpha}{\bar{F}(u)}\right)^{-\xi} - 1\right\}, & \text{for } \xi > 0, \\ u - \sigma(u)\log\left(\dfrac{1-\alpha}{\bar{F}(u)}\right), & \text{for } \xi = 0, \end{cases}$$

where $\bar{F}$ is the survival function of F. We can choose $\sigma(u) = \bar{F}(u)/f(u)$ under the von Mises' condition (see, de Haan and Ferreira (2006)). The relative error of this approximation is given by, with $x = q_\alpha$,

$$\varepsilon_{u,x} := \frac{q_\alpha - \widetilde{q}_\alpha}{q_\alpha} = \begin{cases} 1 - \dfrac{u}{x} - \dfrac{\bar{F}(u)}{\xi x f(u)}\left\{\left(\dfrac{\bar{F}(x)}{\bar{F}(u)}\right)^{-\xi} - 1\right\}, & \text{for } \xi > 0, \quad (1.1) \\ 1 - \dfrac{u}{x} + \dfrac{\bar{F}(u)}{x f(u)}\log\left(\dfrac{\bar{F}(x)}{\bar{F}(u)}\right), & \text{for } \xi = 0. \quad (1.2) \end{cases}$$

for $x > u$. The convergence of this relative error already has been investigated in Beirlant et al. (2003). However, their asymptotic results are in terms of α when u is chosen as a function u_α of α such that $\lim_{\alpha\to 1} F(u_\alpha) = 1$, i.e. they have considered pointwise convergence of $\varepsilon_{u_\alpha,q_\alpha}$ as α tends to one. Therefore, we can not see how the convergence holds true for $q_\alpha > u$ when u tends to infinity. Also, from practical point of view, the threshold u comes first. Therefore, the asymptotic results in terms of α is not so valuable in practice since F is not exactly known, which links u and α.

In this paper, we consider the convergence of the supremum of the relative error

$$S_V(u) := \sup_{x\in(u,V(u))} |\varepsilon_{u,x}|$$

when u tends to infinity, where the function $V(u) : \mathbb{R}^+ \mapsto (u, \infty]$ is a suitably chosen upper bound for the quantile x. Note that the case $V(u) = \infty$ is included in the following propositions and theorems. We will discuss the convergences class by class of the maximum domain of attraction.

2 Relative Error of the Generalized Pareto Approximation to VaR

2.1 In Case of the Fréchet Class

Distribution in the Fréchet class with tail index $\xi > 0$ has the representation

$$\bar{F}(x) = x^{-1/\xi} L(x), \tag{2.1}$$

where $L(x)$ is some slowly varying function, i.e.

$$\lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1 \tag{2.2}$$

for any $t > 0$ (see, e.g. Embrechts et al. (2005)). We hereafter call $L(x)$ a slowly varying factor of F.

Proposition 2.1. *Suppose that F is a distribution function in the Fréchet class with tail index $\xi > 0$ and has a slowly varying factor L. For any $V(u)$, the convergence*

$$\lim_{u \to \infty} S_V(u) = 0 \tag{2.3}$$

is equivalent to

$$\lim_{u \to \infty} \sup_{w \in (1, V(u)/u)} \left| 1 - \left(\frac{L(u)}{L(wu)} \right)^{\xi} \right| = 0. \tag{2.4}$$

Proof. By using the representation in (2.1), we have from (1.1)

$$\varepsilon_{u,wu} = 1 - w^{-1} - \frac{\varphi(u)}{\xi w} \left\{ w \left(\frac{L(u)}{L(wu)} \right)^{\xi} - 1 \right\},$$

where $w = x/u$ and $\varphi(u) = \bar{F}(u)/(uf(u))$. Note that

$$\begin{aligned}
\sup_{w \in (1, V(u)/u)} \left| 1 - \left(\frac{L(u)}{L(wu)} \right)^{\xi} \right| &= \sup_{w \in (1, V(u)/u)} \left| \frac{\xi \varepsilon_{u,wu} + \left(1 - w^{-1}\right)(\varphi(u) - \xi)}{\varphi(u)} \right| \\
&\le \frac{1}{\varphi(u)} \sup_{w \in (1, V(u)/u)} \left[\xi |\varepsilon_{u,wu}| + \left|1 - w^{-1}\right| |\varphi(u) - \xi| \right] \\
&\le \frac{\xi}{\varphi(u)} S_V(u) + \frac{1}{\varphi(u)} |\varphi(u) - \xi|
\end{aligned}$$

for any $u > 0$. Then the convergence in (2.4) follows from (2.3) since the von Mises' condition becomes simpler

$$\lim_{x\to\infty} \varphi(x) = \xi \tag{2.5}$$

in case of the Fréchet class.

The converse is true since

$$\begin{aligned} S_V(u) &\le \sup_{w\in(1,V(u)/u)} \left[\left| \frac{\varphi(u)}{\xi} \left(1 - \left(\frac{L(u)}{L(wu)} \right)^{\xi} \right) \right| + \left| 1 - w^{-1} \right| \left| 1 - \frac{\varphi(u)}{\xi} \right| \right] \\ &\le \frac{\varphi(u)}{\xi} \sup_{w\in(1,V(u)/u)} \left| 1 - \left(\frac{L(u)}{L(wu)} \right)^{\xi} \right| + \frac{|\xi - \varphi(u)|}{\xi}. \end{aligned}$$

□

Proposition 2.1 shows that it is enough to check the convergence in (2.4). The following theorem gives us a choice of $V(u)$ for the convergence.

Theorem 2.2. *Suppose that F is a distribution in the Fréchet class with tail index $\xi > 0$. If*

$$\limsup_{u\to\infty} \frac{V(u)}{u} < \infty, \tag{2.6}$$

then

$$\lim_{u\to\infty} S_V(u) = 0.$$

Proof. There exists some $M > 0$ and $u_0 > 0$ such that

$$\frac{V(u)}{u} < M \text{ for } u > u_0,$$

under the condition(2.6). Then, the equivalent condition (2.4) holds since we have

$$\sup_{w\in(1,V(u)/u)} \left| 1 - \left(\frac{L(u)}{L(wu)} \right)^{\xi} \right| \le \sup_{w\in[1,M]} \left| 1 - \left(\frac{L(u)}{L(wu)} \right)^{\xi} \right|,$$

for any $u > u_0$ and the right side converges to zero as u tends to infinity by the Karamata's convergences theorem (see, e.g. Bingham et al. (1987)). □

Theorem 2.2 gives us a sufficient condition for the convergence of the relative approximation error, but not necessarily sufficient as is seen from the following example.

Example 2.3. We can find a strictly increasing sequence $\{u_n\}_{n\in\mathbb{N}}$ such that $\zeta(u, n) < 1/n$ for $u > u_n$ where

$$\zeta(u, n) = \sup_{w\in(1,n+1)} \left| 1 - \left(\frac{L(u)}{L(wu)} \right)^{\xi} \right|.$$

This is because $\zeta(u,n)$ converges to zero as u tends to infinity for a fixed n by the Karamata's convergence theorem. If we define $V(u)$ as

$$V(u) = \begin{cases} \frac{3}{2}u & \text{for } 0 < u < u_1, \\ (n+1)u & \text{for } u_n \le u < u_{n+1},\ n = 1, 2 \ldots, \end{cases}$$

then $V(u)/u$ diverges to infinity together with u but the condition (2.4) is satisfied.

It is also possible to assure the convergence of $S_V(u)$ by specifying the behaviour of $L(x)$.

Theorem 2.4. *Suppose that F is a distribution in the Fréchet class with tail index $\xi > 0$ and has a slowly varying factor L. If $L(x)$ converges to $l \in (0,\infty)$ as x tends to infinity, then*

$$\lim_{u\to\infty} S_V(u) = 0$$

for any $V(u)$ including $V(u) = \infty$.

Proof. It is clear from the form of (2.4). □

There would be no simple answer to the convergence of $S_V(u)$ when $L(x)$ does not converge to a positive constant. However it is clear that it does not converge to zero as far as $L(x)$ converges to zero or diverges to infinity in case of $V(u) = \infty$.

Proposition 2.1 can be rewritten in terms of the density function f of F. It is known that the density function is written as

$$f(x) = x^{-1/\xi-1} L_f(x)$$

with a slowly varying function L_f as far as F is differentiable and belongs to the Fréchet class with tail index $\xi > 0$ (see, e.g. Klüppelberg et al. (1997)).

Proposition 2.5. *Suppose that F is a distribution function in the Fréchet class with tail index $\xi > 0$ and has a density f which has a slowly varying factor L_f. For any $V(u)$, the convergence*

$$\lim_{u\to\infty} S_V(u) = 0$$

is equivalent to

$$\lim_{u\to\infty} \sup_{w\in(1,V(u)/u)} \left| 1 - \left(\frac{L_f(u)}{L_f(wu)} \right)^{\xi} \right| = 0. \tag{2.7}$$

The proof is placed in Appendix. Theorem 2.4 is now rewritten as the following theorem in terms of the slowly varying factor L_f.

Theorem 2.6. *Suppose that F is a distribution function in the Fréchet class with tail index $\xi > 0$ and has a density f which has a slowly varying factor L_f. If $L_f(x)$ converges to $l \in (0, \infty)$ as x tends to infinity, then*

$$\lim_{u \to \infty} S_V(u) = 0$$

for any $V(u)$ including $V(u) = \infty$.

Proof. The proof is same as that of Theorem 2.4. □

By using this theorem, we can derive a more explicit condition for the choice of $V(u)$ for known densities. We will see it in the following examples.

Example 2.7 (Log-gamma)**.** The slowly varying factor L_f of log-gamma density function is given by

$$L_f(x) = \frac{a^b}{\Gamma(b)} (\log x)^{b-1}, \ (x > 1; \ a > 0, \ b > 0),$$

since the density function is given by

$$f(x) = \frac{a^b}{\Gamma(b)} (\log x)^{b-1} x^{-a-1},$$

and, as is known, log-gamma distribution belongs to the Fréchet class with tail index $1/a$. Therefore, the relative error $S_V(u)$ for any V converges to zero as u tends to infinity when $\beta = 1$ by Theorem 2.6. When $0 < b < 1$ or $b > 1$, the relative error converges to zero as far as $V(u)$ has a representation

$$V(u) = u^{\kappa(u)}, \text{ where } \kappa(u) \searrow 1 \text{ as } u \to \infty.$$

This is because that we have

$$\sup_{w \in (1, V(u)/u)} \left| 1 - \left(\frac{L_f(u)}{L_f(wu)} \right)^{1/a} \right| = \left| 1 - \left(\frac{\log u}{\log V(u)} \right)^{(b-1)/a} \right|.$$

and then a necessary and sufficient condition for the convergence is

$$\lim_{u \to \infty} \frac{\log u}{\log V(u)} = 1,$$

by Proposition 2.1.

Example 2.8 (t distribution)**.** In case of t distribution with the degree of freedom ν, the slowly varying factor L_f is given by

$$L_f(x) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu \pi} \Gamma(\nu/2)} \left(\frac{1}{x^2} + \frac{1}{\nu} \right)^{-(\nu+1)/2}, \ (x \in \mathbb{R}; \ \nu > 0),$$

since the density function is giver by

$$f(x) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\Gamma(\nu/2)} \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2},$$

and the t distribution is element of the Fréchet class with tail index $1/\nu$. The relative approximation error $S_V(u)$ with $V = \infty$ converges to zero as u tends to infinity, since the $L_f(x)$ has a positive limiting constant as x tends to infinity.

The same discussion follows as in Example 2.8 for inverse-gamma, F, Cauchy distributions. Therefore, it is free to use the generalized Pareto approximation to log-gamma with $b = 1$, t, inverse-gamma, F or Cauchy distributions.

2.2 In Case of the Gumbel Class

We investigate the relative approximation error in case of the Gumbel class distributions. We restrict our attention into a sub-class, Weibull-type of the Gumbel distributions.

Definition 2.9 (Weibull-type)**.** *F is a distribution function of Weibull-type with index $\beta \geq 0$ if $\bar{F}$ is written as*

$$\bar{F}(x) = \exp\left(-x^\beta L(x)\right),$$

with a differentiable slowly varying function $L(x)$ which satisfies the following condition,

- $\lim_{x\to\infty} \frac{xL'(x)}{L(x)} = 0$ *in case of* $\beta > 0$,

- $\lim_{x\to\infty} L(x) = \infty$ *in case of* $\beta = 0$,

wehere L' is derivative of L.

This class includes, for instance, normal distribution ($\beta = 2$), Gamma distribution ($\beta = 1$) and log-normal distribution ($\beta = 0$).

From the form of density function of Weibull-type distribution

$$f(x) = \begin{cases} \left(\beta x^{\beta-1} L(x) + x^\beta L'(x)\right) \exp\left(-x^\beta L(x)\right), & \beta > 0, \\ L'(x)\exp(-L(x)), & \beta = 0, \end{cases}$$

the relative approximation error (1.2) can be written as, with $w = x/u$,

$$\varepsilon_{u,wu} = \begin{cases} 1 - w^{-1} + w^{-1}\dfrac{L(u) - w^\beta L(wu)}{\beta L(u) + uL'(u)}, & \text{for } \beta > 0, \quad (2.8) \\ 1 - w^{-1} + w^{-1}\dfrac{L(u) - L(wu)}{uL'(u)}, & \text{for } \beta = 0\ . \quad (2.9) \end{cases}$$

Proposition 2.10. *Suppose that F is a Weibull-type distribution with index $\beta > 0$ and has a slowly varying factor L. For any $V(u)$, the convergence*

$$\lim_{u\to\infty} S_V(u) = 0 \tag{2.10}$$

is equivalent to

$$\lim_{u\to\infty} \sup_{w\in(1,V(u)/u)} \left| 1 + \frac{1-\beta}{\beta} w^{-1} - \frac{w^{\beta-1}}{\beta} \frac{L(wu)}{L(u)} \right| = 0. \tag{2.11}$$

Proof. The convergence in (2.11) follows from (2.10) since, with $\varphi(u) = uL'(u)/L(u)$,

$$\begin{aligned}
&\sup_{w\in(1,V(u)/u)} \left| 1 + \frac{1-\beta}{\beta} w^{-1} - \frac{w^{\beta-1}}{\beta} \frac{L(wu)}{L(u)} \right| \\
&\leq \sup_{w\in(1,V(u)/u)} \left[\left| \left(1 + \frac{\varphi(u)}{\beta}\right) \varepsilon_{u,wu} \right| + \left| \frac{\varphi(u)}{\beta} \left(w^{-1} - 1\right) \right| \right] \\
&\leq \left(1 + \frac{|\varphi(u)|}{\beta}\right) S_V(u) + \frac{|\varphi(u)|}{\beta}
\end{aligned}$$

from (2.8) and we have $\lim_{u\to\infty} \varphi(u) = 0$ by Definition 2.9.

In a similar way, the converse can be proved since

$$\begin{aligned}
S_V(u) &\leq \left| 1 + \frac{\varphi(u)}{\beta} \right|^{-1} \sup_{w\in(1,V(u)/u)} \left[\left| 1 + \frac{1-\beta}{\beta} w^{-1} - \frac{w^{\beta-1}}{\beta} \frac{L(wu)}{L(u)} \right| \right. \\
&\qquad\qquad \left. + \left| \frac{\varphi(u)}{\beta} \left(w^{-1} - 1\right) \right| \right] \\
&\leq \left| 1 + \frac{\varphi(u)}{\beta} \right|^{-1} \left[\sup_{w\in(1,V(u)/u)} \left| 1 + \frac{1-\beta}{\beta} w^{-1} - \frac{w^{\beta-1}}{\beta} \frac{L(wu)}{L(u)} \right| + \frac{|\varphi(u)|}{\beta} \right].
\end{aligned}$$

□

2.2.1 Weibull type with $\beta = 1$

The following theorems give us a sufficient condition for the convergence of the relative error. These conditions are same as in the case of the Fréchet class.

Theorem 2.11. *Suppose that F is a Weibull-type distribution with index $\beta = 1$ and has a slowly varying factor L. If*

$$\limsup_{u\to\infty} \frac{V(u)}{u} < \infty, \tag{2.12}$$

then

$$\lim_{u\to\infty} S_V(u) = 0.$$

Proof. From (2.11), the equivalent condition is

$$\lim_{u\to\infty} \sup_{w\in(1,V(u)/u)} \left|1 - \frac{L(wu)}{L(u)}\right| = 0,$$

in case of $\beta = 1$. It holds true for any $V(u)$ which satisfies the condition (2.12), by the Karamata's convergence theorem as in the proof of Theorem 2.2. □

Note that this is only a sufficient condition as same as in Example 2.3.

Theorem 2.12. *Suppose that F is a Weibull-type distribution with index $\beta = 1$ and the slowly varying factor L. If the slowly varying factor $L(x)$ converges to $l \in (0,\infty)$ as x tends to infinity, then the equivalent condition (2.11) holds true for any $V(u)$.*

Proof. The proof is same as in Theorem 2.4 with $\xi = 1$. □

2.2.2 Weibull type with $\beta \neq 1$

In this case, we can derive a necessary and sufficient condition for the convergence of the relative error to zero. The uniform convergence is assured only in the neighborhood of the threshold u, while the convergence of $S_\infty(u)$ is possible in other case.

Theorem 2.13. *Suppose that F is a Weibull-type distribution with index $\beta \neq 1$ and has a slowly varying factor L. For any $V(u)$,*

$$\lim_{u\to\infty} S_V(u) = 0,$$

if and only if

$$\lim_{u\to\infty} \frac{V(u)}{u} = 1. \tag{2.13}$$

Proof. We have to distinguish the case $0 < \beta \neq 1$ and the case $\beta = 0$.

1. In case of $0 < \beta \neq 1$.

   First, we show that the convergence of $S_V(u)$ to zero never holds as u tends to infinity if $\limsup_{u\to\infty} V(u)/u > 1$. For all $u_n > 0$ $(n = 1, 2 \ldots)$ with $\lim_{n\to\infty} u_n = \infty$, there exists some $u_n^* > u_n$ and $\delta_n > 0$ with $\lim_{n\to\infty} \delta_n = \delta > 0$ such that

   $$\frac{V(u_n^*)}{u_n^*} > 1 + \delta_n.$$

   Then, for each n,

   $$\begin{aligned}
   &\sup_{w\in(1,V(u_n^*)/u_n^*)} \left|1 + \frac{1-\beta}{\beta} w^{-1} - \frac{1}{\beta} w^{\beta-1} \frac{L(wu_n^*)}{L(u_n^*)}\right| \\
   &\qquad \geq \sup_{w\in(1,1+\delta_n)} \left|1 + \frac{1-\beta_n}{\beta} w^{-1} - \frac{1}{\beta} w^{\beta-1} \frac{L(wu_n^*)}{L(u_n^*)}\right| \\
   &\qquad \geq \left|1 + \frac{1-\beta}{\beta}(1+\delta_n)^{-1} - \frac{1}{\beta}(1+\delta_n)^{\beta-1} \frac{L((1+\delta_n)u_n^*)}{L(u_n^*)}\right|.
   \end{aligned}$$

The last expression converges to

$$1 + \frac{1-\beta}{\beta}(1+\delta)^{-1} - \frac{1}{\beta}(1+\delta)^{\beta-1}$$

as n increases, which is strictly positive by the Karamata's convergence theorem. From Proposition 2.10, $S_V(u)$ never converges to zero as u tends to infinity.

To complete the proof, we have only to show that if $\lim_{u\to\infty} V(u)/u = 1$ then $S_V(u)$ converges to zero as u tends to infinity. From Proposition 2.10, it is enough to show

$$\lim_{u\to\infty} \sup_{w\in(1,V(u)/u)} \left| 1 + \frac{1-\beta}{\beta} w^{-1} - \frac{1}{\beta} w^{\beta-1} \frac{L(wu)}{L(u)} \right| = 0.$$

Note that

$$\begin{aligned}
&\sup_{w\in(1,V(u)/u)} \left| 1 + \frac{1-\beta}{\beta} w^{-1} - \frac{1}{\beta} w^{\beta-1} \frac{L(wu)}{L(u)} \right| \\
&= \sup_{w\in(1,V(u)/u)} \left[\left| 1 + \frac{1-\beta}{\beta} w^{-1} - \frac{1}{\beta} w^{\beta-1} \right| + \frac{1}{\beta} w^{\beta-1} \left| \left(1 - \frac{L(wu)}{L(u)} \right) \right| \right] \\
&\leq \left| 1 + \frac{1-\beta}{\beta} \frac{u}{V(u)} - \frac{1}{\beta} \left(\frac{u}{V(u)} \right)^{1-\beta} \right| \\
&\qquad + \frac{1}{\beta} \left(\sup_{w\in(1,V(u)/u)} w^{\beta-1} \right) \left(\sup_{w\in(1,V(u)/u)} \left| 1 - \frac{L(wu)}{L(u)} \right| \right).
\end{aligned}$$

Each terms in the last expression converges to zero as u tends to infinity by the Karamata's convergence theorem as far as $\limsup_{u\to\infty} V(u)/u < \infty$ and the condition (2.13) holds.

2. In case of $\beta = 0$.

   We have that

$$\begin{aligned}
&\sup_{w\in(1,V(u)/u)} \left| 1 - w^{-1} - w^{-1} \frac{L(wu) - L(u)}{uL'(u)} \right| \\
&\leq \sup_{w\in(1,V(u)/u)} \left[\left| 1 - w^{-1} - w^{-1} \log w \right| + \left| \frac{L(wu) - L(u)}{uL'(u)} - \log w \right| \right] \\
&= \left| 1 - \frac{u}{V(u)} - \frac{u}{V(u)} \log\left(\frac{V(u)}{u} \right) \right| + \sup_{w\in(1,V(u)/u)} \left| \frac{L(wu) - L(u)}{uL'(u)} - \log w \right|,
\end{aligned}$$

   from (2.9). For the second term in the last expression, we have

$$\lim_{u\to\infty} \frac{L(wu) - L(u)}{uL'(u)} = \lim_{u\to\infty} \int_1^w \frac{L'(tu)}{L'(u)} dt,$$

and

$$\lim_{u\to\infty} \frac{L'(tu)}{L'(u)} = t^{-1},$$

from (2.2). Therefore, the relative error $S_V(u)$ converges to zero as u tends to infinity if the condition (2.13) holds.

Because of the same argument in the case $0 < \beta \neq 1$, the convergence of $S_V(u)$ to zero never holds if $\lim_{u\to\infty} V(u)/u > 1$.

□

2.3 Comparison between t and normal distributions

It is interesting to compare t distribution and normal distribution. As is seen in Example 2.8, t distribution belongs to the Fréchet class and normal distribution is a Weibull-type distribution with $\beta = 2$. Therefore, the uniform convergence is assured for any choice of $V(u)$ for t distribution, but it does not holds true for normal distribution. We will see what happens when the degree of freedom of t distribution tend to infinity. To do this, we have to come back to Proposition 2.5. The convergence of $S_V(u)$ to zero is equivalent to

$$\lim_{u\to\infty} \sup_{w\in(1,V(u)/u)} \left| 1 - \left(\frac{L_f(u)}{L_f(wu)} \right)^{1/\nu} \right| = 0,$$

where

$$L_f(x) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\,\Gamma(\nu/2)} \left(\frac{1}{x^2} + \frac{1}{\nu} \right)^{-(\nu+1)/2}.$$

Then, we get

$$\sup_{w\in(1,V(u)/u)} \left| 1 - \left(\frac{L_f(u)}{L_f(wu)} \right)^{1/\nu} \right| = \sup_{w\in(1,V(u)/u)} \left| 1 - \left(\frac{1 + w^2u^2\nu^{-1}}{w^2 + w^2u^2\nu^{-1}} \right)^{1/2+2\nu} \right|$$

Since the right hand side converges to $1 - u/V(u)$ as ν tends to infinity, the condition

$$\lim_{u\to\infty} \frac{V(u)}{u} = 1$$

follows as ν tends to infinity.

We have seen that uniform convergence of the relative error is not always assured. The convergence heavily depends on the shape of the distribution $F(x)$, particularly its slowly varying factor $L(x)$. In the next section, we will see how well the approximation holds true in practice by some of computer simulations.

3 Numerical Result

In this section we report the results of Monte Carlo simulations. We have generated 100000 i.i.d. random numbers in each experiment. In the following, each panel show a Quantile-Quantile plot of the GPD quantile against the sample quantile of the excess distribution.

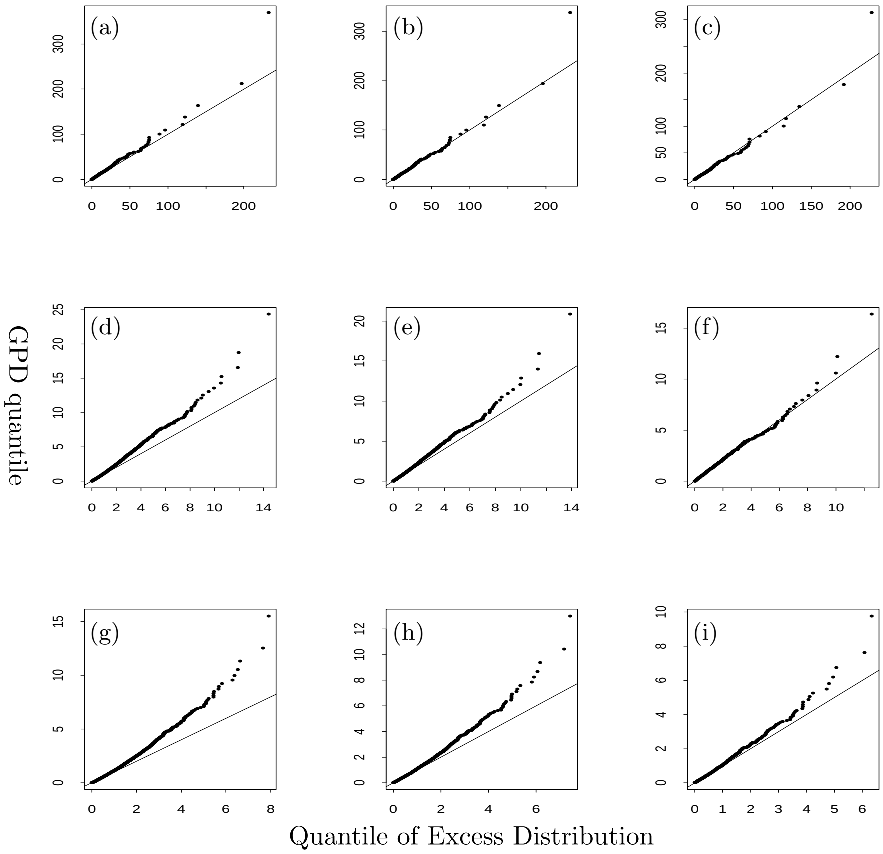

Figure 1: Quantile-Quantile plots in case of the t distribution.

Figure 1 shows the results for the case of t distributions. The parameters of the GPD are taken to be $\xi = 1/\nu$ and $\sigma = (1 - t_\nu(u))/t'_\nu(u)$ for degrees of freedom ν, where t_ν is the distribution function of the t distribution. The degrees of freedom are the same for each row of the panel matrix (2, 5 and 7 respectively from top to bottom). The threshold is the same for each column ($t_\nu^{-1}(0.9)$, $t_\nu^{-1}(0.95)$ and $t_\nu^{-1}(0.99)$ respectively from left to right). We can see from the panels that the approximation works well for ν small, but does not work well if ν is large and u is small.

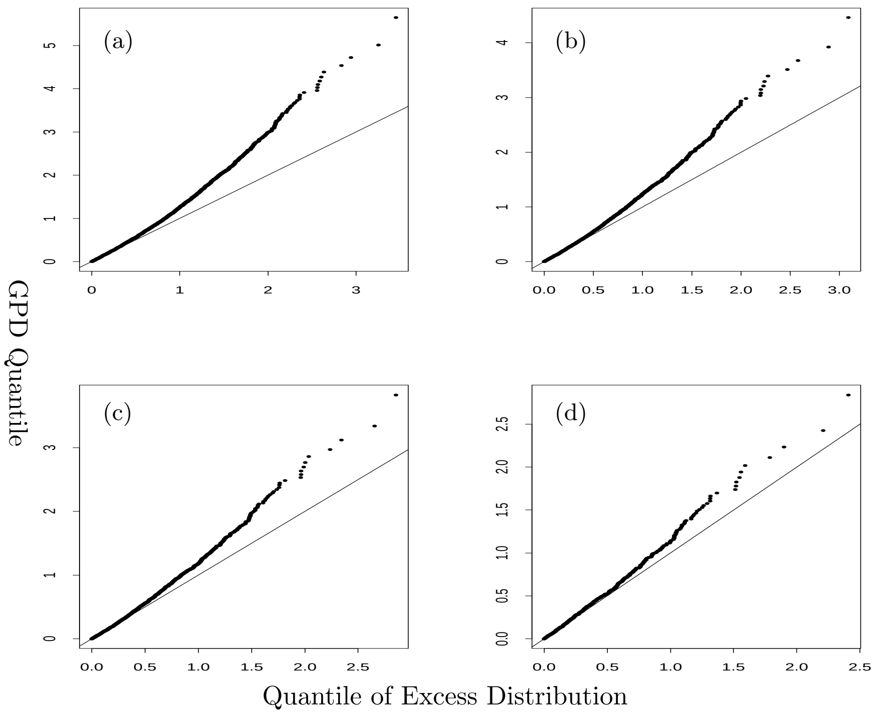

Figure 2: Quantile-Quantile plots in case of the normal distribution.

Figure 2 shows the results for the case of the normal distribution. The parameters of the GPD are taken to be $\xi = 0$ and $\sigma = (1 - \Phi(u))/\Phi'(u)$ where Φ is the normal distribution function. The threshold u is taken to be $\Phi^{-1}(0.9)$, $\Phi^{-1}(0.95)$, $\Phi^{-1}(0.97)$ and $\Phi^{-1}(0.99)$ respectively from (a) to (d). The approximation does not work well for any u, except in the neighborhood of u. These results are consistent with the theoretical results given in Section 2.

4 Concluding Remarks

Our result shows that the generalized Pareto approximation does not always provide a good estimate of the quantile $x = F^{-1}(\alpha)$, $x > u$, although it gives a good approximation to the excess distribution function $F_u(x-u)$ itself. Caution is therefore needed in applications. It is safe to restrict attention to $u < x < V(u)$ such that $\lim_{u\to\infty} V(u)/u = 1$. Therefore, a practical procedure of VaR_α estimation is to find an appropriate u where there are significant number of observations on the left but not so many on the right.

Appendix

Proof of Proposition 2.5
By the von Mises' condition (2.5), for any $\varepsilon \in (0, \xi)$ there exists some $u_0 > 0$ such that

$$\left| \frac{L(u)}{L_f(u)} - \xi \right| < \varepsilon$$

for any $u > u_0$. Then, we have

$$\lim_{u\to\infty} \sup_{w\in(1,V(u)/u)} \left| \left(\frac{L(u)/L_f(u)}{L(wu)/L_f(wu)} \right)^{\xi} - 1 \right| = 0,$$

since

$$\begin{aligned} &\sup_{w\in(1,V(u)/u)} \left| \frac{L(u)/L_f(u)}{L(wu)/L_f(wu)} - 1 \right| \\ &\quad \le \sup_{w\in(1,V(u)/u)} \left[\frac{|L(u)/L_f(u) - \xi| + |L(wu)/L_f(wu) - \xi|}{L(wu)/L_f(wu)} \right] < \frac{2\varepsilon}{\xi - \varepsilon}, \end{aligned}$$

for any $u > u_0$. Therefore, the equivalent condition (2.4) in Proposition 2.1 follows from (2.7) since

$$\begin{aligned} &\sup_{w\in(1,V(u)/u)} \left| 1 - \left(\frac{L(u)}{L(wu)} \right)^{\xi} \right| \\ &= \sup_{w\in(1,V(u)/u)} \left| 1 - \left(\frac{L_f(u)}{L_f(wu)} \right)^{\xi} - \left\{ \left(\frac{L(u)/L_f(u)}{L(wu)/L_f(wu)} \right)^{\xi} - 1 \right\} \left(\frac{L_f(u)}{L_f(wu)} \right)^{\xi} \right| \\ &\le \sup_{w\in(1,V(u)/u)} \left| 1 - \left(\frac{L_f(u)}{L_f(wu)} \right)^{\xi} \right| \\ &\qquad + \sup_{w\in(1,V(u)/u)} \left| \left(\frac{L(u)/L_f(u)}{L(wu)/L_f(wu)} \right)^{\xi} - 1 \right| \sup_{w\in(1,V(u)/u)} \left(\frac{L_f(u)}{L_f(wu)} \right)^{\xi}. \end{aligned}$$

The converse holds true since

$$\begin{aligned} &\sup_{w\in(1,V(u)/u)} \left| 1 - \left(\frac{L_f(u)}{L_f(wu)} \right)^{\xi} \right| \\ &\quad \le \sup_{w\in(1,V(u)/u)} \left| 1 - \left(\frac{L(u)}{L(wu)} \right)^{\xi} \right| \\ &\qquad + \sup_{w\in(1,V(u)/u)} \left| \left(\frac{L(wu)/L_f(wu)}{L(u)/L_f(u)} \right)^{\xi} - 1 \right| \sup_{x\in(u,V(u))} \left(\frac{L(u)}{L(wu)} \right)^{\xi}. \end{aligned}$$

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